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The random normal superconductor mixture in one dimension

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Abstract. Two models for random two-conductor mixtures by diffusion processes are considered in the one-dimensional case. It is shown that the two models are not in the same universality class. This is seen to be an artefact of systems with $p_c = 1$, however. A scaling theory proposed earlier is tested for this simple one-dimensional case and confirmed.

1. Introduction

In the spirit of de Gennes' idea to study conductivity problems by investigating a corresponding diffusion process (de Gennes 1980), the conductivity of a random mixture of normal metal and superconductor has been modelled by various diffusion processes (Coniglio and Stanley 1984, Bunde *et al* 1985, Adler *et al* 1985, Hong *et al* 1985). It is the purpose of this paper to investigate these various processes in the one-dimensional case, where all the exponents can be exactly obtained. The most striking fact we notice is that two models that were both proposed to model the random superconducting normal mixture are in different universality classes. This peculiarity can, however, easily be explained and should not persist in higher dimensions, though it will presumably occur in any structure with $p_c = 1$. Further, we obtain various scaling exponents and thereby confirm predictions following from Hong *et al* (1985). Similar work on one dimension was recently done by Sahimi and Siddiqui (1985), though their numerical results disagree with the findings of Hong *et al* (1985).

The two models proposed are as follows.

(*a*) The 'short-circuit termite', introduced by Coniglio and Stanley (1984) and extensively developed by Adler *et al* (1985) as the 'Tel-Aviv termite'. In this model the random walker walks in the usual way on normal sites, using a time τ_B per step. As soon as it steps on a superconducting site, however, it can jump to any other superconducting site in the same cluster by means of instantaneous long-range jumps, as well as leave it for a neighbouring normal site. For a more detailed description of the algorithm, see Adler *et al* (1985).

(*b*) The 'two-time termite' also introduced by Coniglio and Stanley (1984) and developed as the 'Boston termite: model I' by Bunde *et al* (1985) is defined as follows. Consider a mixture of two components with conductivities σ_A, σ_B such that

$$h = \sigma_B/\sigma_A \ll 1. \tag{1}$$

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Now define a random walker that has two time steps τ_A and τ_B on each component A or B with τ_A and τ_B proportional to σ_A and σ_B respectively. If, further, the walker is assumed to move with a likelihood greater by a factor of h^{-1} from B to A than from A to B, then it is readily seen that this should describe a mixture of two conductors with conductivities σ_A and σ_B. We now proceed to investigate the properties of these random walks in the one-dimensional case.

2. The mean square distance

The 'short-circuit' termite does not spend any time inside a superconducting cluster. We can thus replace every superconducting cluster by a single 'renormalised' bond of length unity. The result is nearest-neighbour diffusion on a one-dimensional chain with a random distribution of two types of bonds. The random walker (termite) on this chain has the usual properties, i.e. the mean square distance L is given by

$$\langle L^2 \rangle \propto \text{constant} \times t/\tau_B \tag{2}$$

where t is the number of steps. The renormalised length L_0 of the chain is of the order of the number of 'normal' sites in the original chain. Therefore, if the original separation between two points was r, the renormalised distance L_0 will be given by

$$L_0 \propto (1-p)r. \tag{3}$$

Combining (1) and (2) one finds

$$\langle r^2 \rangle \propto t/[(1-p)^2 \tau_B]. \tag{4}$$

Hence the diffusion coefficient $D(p)$ scales as $(1-p)^{-2}$. We emphasise that equation (2) is true for all $t \geqslant 1$ and therefore the result (4) also applies for all $t \geqslant 1$. Note that if each normal site represented a resistance 1, then the average resistance of a sample of size r would scale as $L \propto (1-p)r$ and the conductivity $\sigma = [\text{length}]/[\text{resistance}]$ scales as

$$\sigma \propto (1-p)^{-1} \tag{5}$$

i.e. the exponent s, defined as $\sigma \propto \varepsilon^{-s}$, is equal to 1. Thus we find that

$$\sigma \propto (1-p)D(p). \tag{6}$$

Indeed, the Einstein relation demands that the ratio σ/D be proportional to the density of charge carriers n and the 'short-circuit' algorithm only involves charge carriers (i.e. 'termites') on the *normal* sites, so that $n \propto (1-p)$. The extra factor of $(p_c - p)$ is thus peculiar to systems with $p_c = 1$ (one dimension, Sierpinski gasket, etc).

For the two-time termite, however, n is not critical in one dimension since the termite can walk everywhere. Thus we expect $D_{TT}(p) \propto (1-p)^{-1}$, where the subscript TT denotes the 'two-time termite'. One can derive more rigorously this critical behaviour. Suppose the average termite density is ρ_A and ρ_B in A and B clusters respectively. Let the transition probability from B to A be W_{BA} and A to B W_{AB}. Then at the boundary a detailed balance condition should be satisfied:

$$\rho_A W_{AB} = \rho_B W_{BA}. \tag{7}$$

But $\rho_A \propto N_A/V_A$ and $\rho_B \propto N_B/V_B$ where N_A and N_B are the number of steps taken by the termite on A and B sites and V_A and V_B stand for the volume of the A and B clusters. Moreover, $W_{BA}/W_{AB} = \sigma_A/\sigma_B$ and $V_A/V_B = p/(1-p)$. Thus we have

$$(N_A\sigma_A^{-1})/(N_B\sigma_B^{-1}) = p/(1-p) \tag{8}$$

or equivalently

$$t_A/t_B = p/(1-p) \tag{9}$$

where t_A and t_B are the actual times spent by the termite. Using equation (1) and $t = t_A + t_B = N_A\tau_A + N_B\tau_B$, one can thus have

$$D_{TT}(\sigma_B, \sigma_A, p) = D_{SC}(\sigma_B, \sigma_A, p)(1-p) \tag{10}$$

where $D_{SC}(p)$ is the diffusion constant of the short-circuit model.

Thus we have

$$D_{TT}(p) \propto (1-p)^{-1}. \tag{11}$$

3. The first exit time

Previous work (Coniglio and Stanley 1984, Sahimi and Siddiqui 1985, Adler *et al* 1985) had considered at length the 'anomalous diffusion' of termites for 'short' times, i.e. $h \ll t/\tau_B \ll 1$. In the short-circuit model, however, this would be impossible, since diffusion over a superconducting cluster is instantaneous. Using a different average, however, one can obtain such a short-time behaviour; we worked at fixed r and averaged over the variations of the first exit time. The first exit time T is defined as the first time the termite travels a distance larger than r, i.e. at time T the distance is $\leqslant r$ and at time $T+1$ it is $> r$. Clearly, if the origin lies in the middle of a superconducting cluster which contains more than $2r$ sites, T will be equal to zero; the clock does not advance until the termite reaches the edge of the cluster.

We note at this point that one of the algorithms we will discuss below differs slightly from that presented in Adler *et al* (1985). In the old algorithm the termite was initially parachuted onto any lattice site $x = 0$ and then chose one of that site's nearest neighbours. In the new algorithm it is again parachuted onto any site; however, only if the site chosen is a normal site does it immediately choose a neighbouring site. If it is initially parachuted onto a superconducting site in the new algorithm it first chooses another site of the superconducting cluster at random and only then proceeds to choose a neighbouring site. In both algorithms the procedure after the neighbouring site is chosen is the same, i.e. if it is a normal site the clock advances by one time step, whereas if it is a superconducting site it chooses another site of the same cluster at random. It then again looks at the neighbouring sites. The difference between the two algorithms is trivial for large r and t values but is considerable for small r values in one dimension, as we now proceed to demonstrate.

For the case $1 \ll r \ll (1-p)^{-1}$, we can estimate T for both algorithms. If there are no normal sites between r and $-r$, then clearly $T = 0$ in both cases. If there is one normal site between r and $-r$, then *if* this site is adjacent to the site at $x = 0$ onto which the termite was dropped *and if* we are using the orignal algorithm of Adler *et al* (1985), wherein a termite parachuted on a superconducting site can immediately

choose a neighbouring site, we have a contribution of order $(1-p)\tau_B$. If, on the other hand, the normal site is situated at $1<|x|\leq r$ or if we are using the new algorithm and $|x|=1$ or if the termite is parachuted onto a normal site, the situation is quite different: the termite will leave the interval between r and $-r$ by the first jump (and therefore at $T=0$, since the clock does not advance when the termite *leaves* a normal site, but only when it *steps* onto one) with probability $1-(r+|x|)/S$, where S is the expected number of sites on a cluster, given that one already is on a cluster. For an infinite system, using the fact that p is very close to one, one has $S=2p/(1-p)+1\approx 2/(1-p)$. If the termite does not exit by the first jump, then the situation is rather more complex. With probability $1/S$ it will immediately jump to the site adjoining x and since there are of the order of $2r(1-p)$ configurations of this type, it will thus give a contribution of order $(1-p)^2r\tau_B$ to $\langle T\rangle$. With probability $(|x|+r-1)/S$ it will immediately jump to one of the other sites on the cluster within the $[-r, r]$ interval and from here with probability $1/S$ it will then jump to the site adjacent to x, thus giving a contribution of order $[(|x|+r-1)(1-p)^3r/2]\tau_B$ and so on. In the limit $(1-p)r\ll 1$ these terms can be summed since $|x|-1$ is of order r, to give a total contribution of order (we neglect $1-p$ compared to one)

$$\frac{(1-p)^2r}{1-(1-p)r}\,\tau_B. \tag{12}$$

If there are two normal sites between r and $-r$—this happens with probability $(1-p)^2r^2$—then $\langle T\rangle$ is of the order of unity if those sites are on opposite sides of $r=0$, and thus we have

$$T/\tau_B \propto a_1(1-p)+a_2(1-p)^2r+a_3(1-p)^2r^2 \tag{13}$$

where a_1 is zero in the new algorithm. In our limit the second term will be much smaller than the first for the original algorithm of Adler *et al* (1985), since $(1-p)^2r\ll 1-p$, and it will also be much smaller than the third term, since $(1-p)^2r\ll(1-p)^2r^2$. Thus for both algorithms $\langle T\rangle\ll 1$, and for the original algorithm of Adler *et al* (1985) $\langle T\rangle$ will be essentially independent of r with correction terms going as r^2 and r. For the new algorithm $\langle T\rangle$ goes as r^2 with correction terms going as r.

For very large $(1-p)r$, on the other hand, the termite must perform a random walk through many normal sites before it reaches a distance r. In this limit we can thus return to the renormalised chain of § 2 and argue that

$$T/\tau_B \propto L^2 \propto (1-p)^2r^2. \tag{14}$$

Thus $\langle T\rangle$ scales in the same way as t for long distances, but differs from it for short ones in the original algorithm of Adler *et al* (1985).

We can calculate exact lower bounds for $\langle T\rangle$ corresponding to small values of r from an asymptotic expansion in the limit where p is close to one. In order to do this, we must consider different configurations of normal and superconducting sites and the contribution of each to the first exit time. The different contributions that we will discuss in detail are listed in figure 1. These results complement those presented above for the limit $1\ll r\ll(1-p)^{-1}$; in this case we do not require $(1-p)r\ll 1$ and r is in fact of the order of one. We note that while this calculation does give a lower bound for all values of p and r, it is only useful for small r and large p. The contributions to $\langle T\rangle$ from all the configurations listed in figure 1 are given in table 1. Configurations with one normal site are called C, two normal sites D and three E. We consider terms

(*a*)		−4	−3	−2	−1	0	1	2	3	4
$r=1$	C1	·	·	·	●	○	●	·	·	·
	C2	·	·	·	○	●	●	·	·	·
$r=2$	C1	·	·	●	●	○	●	●	·	·
	C2	·	·	●	○	●	●	●	·	·
	C3	·	·	○	●	●	●	●	·	·
$r=3$	C1	·	●	●	●	○	●	●	●	·
	C2	·	●	●	○	●	●	●	●	·
	C3	·	●	○	●	●	●	●	●	·

(*b*)		−4	−3	−2	−1	0	1	2	3	4
$r=1$	D1	·	·	·	○	●	○	·	·	·
	D2	·	·	·	○	○	●	·	·	·
$r=2$	D1	·	·	○	●	●	●	○	·	·
	D2	·	·	●	○	○	●	●	·	·
	D3	·	·	○	○	●	●	●	·	·
$r=3$	D1	·	●	○	●	●	●	○	●	·
	D2	·	●	○	●	○	●	●	●	·
	D3	·	○	●	○	●	●	●	●	·
	D4	·	○	○	●	●	●	●	●	·

Figure 1. (*a*) Configuration of superconductor and normal sites for $r=1$, 2 and 3 that give contributions of order $p^{2r}(1-p)$ to T. Note that those configurations that are not symmetric about the origin are only listed with the normal sites to the left, and that C3 includes all cases where there is a normal site at $|x|>1$. (*b*) Configuration of superconductor and normal sites that give contributions of order $p^{2r-1}(1-p)^2$ to T. Not all configurations are listed explicitly; for those that give identical contributions only one representative has been chosen. For example, D1 includes all cases where one normal site is at x, $x>0$ and the other at x', $x'<0$, D2 includes all cases where one normal site is at the origin, D3 includes all cases where one normal site is at $x=\pm 1$ and the other at x' where x and x' are on the same side of the origin and D4 is representative of cases where both normal sites are at $|x|>1$ and to the same side of $x=0$. (●, superconductor, ○, normal, ·, either superconductor or normal.)

Table 1.

$r=1$	Occurrence	Probability of $T=\tau_B$
C1	$p^{2r}(1-p)$	$(S-1)/2S^2$
C2	$2p^{2r}(1-p)$	$(S-1)/S^2$
C3	$2(r-1)p^{2r}(1-p)$	$(S-1)/2S^2$
D1	$[r+r(r-1)]p^{2r-1}(1-p)^2$	2
D2	$2rp^{2r-1}(1-p)^2$	1
D3	$2(r-1)p^{2r-1}(1-p)^2$	—
D4	$(r-2)(r-1)p^{2r-1}(1-p)^2$	—

of highest orders in p (lowest order in $1-p$) and assume that contributions of higher order can be neglected in the limit when r and p go to one. We discuss the bounds in detail for the new algorithm only.

Let us illustrate the calculation of the bounds with two examples, the cases of $r=1$, configuration C1 and C2. We wish to calculate the time taken to exit from the three sites, i.e. to reach $|x|>1$. The termite is parachuted onto the origin at $T=0$. On configuration C1 it then moves to $x=\pm 1$ with equal probability. Since the sites at ± 1 belong to superconducting clusters of expected size S, it then has to choose one site of the cluster at random. With probability $(S-1)/S$ it moves to $|x|>1$ at this step and thus has $r>1$ with no contribution to T. However, with probability $1/S$ it can stay at ± 1, and then has a probability of $\frac{1}{2}$ of returning to the origin at the next step. As it now goes onto a normal site, $T=1$. It then returns to ± 1 and with probability $(S-1)/S$ exits when it chooses a random site of the cluster. Thus we have $T=1$ for C1 with probability at least $(S-1)/S\times\frac{1}{2}\times(1/S)$ and we have probability $p^2(1-p)$ of configuration C1 occurring. Configuration C2 occurs with probability $2p^2(1-p)$ and gives a contribution to $T=1$ of at least $(1/S)\times\frac{1}{2}\times\frac{1}{2}$ from the path $0\to 0\to -1\to x<-1$ and of at least $(1/S)\times\frac{1}{2}\times\frac{1}{2}\times(S-2)/S$ from the path $0\to 0\to -1\to 0\to x>1$. The contributions listed above are the lowest order terms in $1/S$ for C1 and C2. All other paths on C1 and C2 are of order $1/S^2$ and can thus be neglected in the asymptotic expansion for small r.

The contributions from C1, C2, C3 and C4 for other values of r can be estimated by similar considerations. The D1 and D2 configurations for cases where the occurrence has a term of order $(1-p)^2$ give a much larger contribution per term but occur far less frequently. For the limit $1\ll r\ll(1-p)^{-1}$ these terms dominated the C type configurations by a factor of r; here their contributions are of a similar order of magnitude, but still dominate in most cases. As an example we consider the case $r=1$ configuration D1. The termite will take the path $0\to\pm 1\to|x|>1$ with probability $\frac{1}{2}$, giving a contribution of $\frac{1}{2}$ to $\langle T\rangle$, the path $0\to\pm 1\to 0\to\pm 1\to|x|>1$ with probability $\frac{1}{4}$, giving a contribution of $\frac{2}{4}$ to $\langle T\rangle$ (since $T=2$ for this path) and the path $0\to\pm 1\to 0\to\pm 1\to 0\to\pm 1\to|x|>1$ with probability $\frac{1}{8}$ giving a contribution of $\frac{3}{8}$ and so on. Thus $\langle T\rangle=\sum_{k=0}^{\infty}k(\frac{1}{2})^k=2$. For the D2 configuration the sum is half the D1 value since there is an initial probability of $\frac{1}{2}$ for the termite to exit on a path with $T=0$. For the D3 and D4 configurations the contribution to $\langle T\rangle$ is not of order unity, since with probability of order $1-$ constant $\times 1/S$ the termite exits before reaching a normal site. We do not discuss the E configurations in any detail.

We note briefly that it is also possible to give exact lower bounds for the old algorithm. Here the leading terms are $p^2(1-p)(S-1)/S$, $p^4(1-p)(S-2)/2$, and $p^6(1-p)(S-3)/S$ for $r=1$, 2 and 3 respectively; these come from configurations C2. In the old algorithm, the D3 contributions differ from the D4 ones, being of the same order as the D2 terms, but all D terms are of higher order in $1-p$ and the same order in $1/S$ as the C2 ones and could thus be neglected as $p\to 1$ for very small r. For larger r both must be considered.

4. Numerical results: the short-circuit

We have carried out more extensive calculations of the first exit time T for termites in one dimension, and also tested equation (11). A graph of first exit times as a function of r^2 for the original algorithm of Adler *et al* (1985) is presented in figure 2; this graph

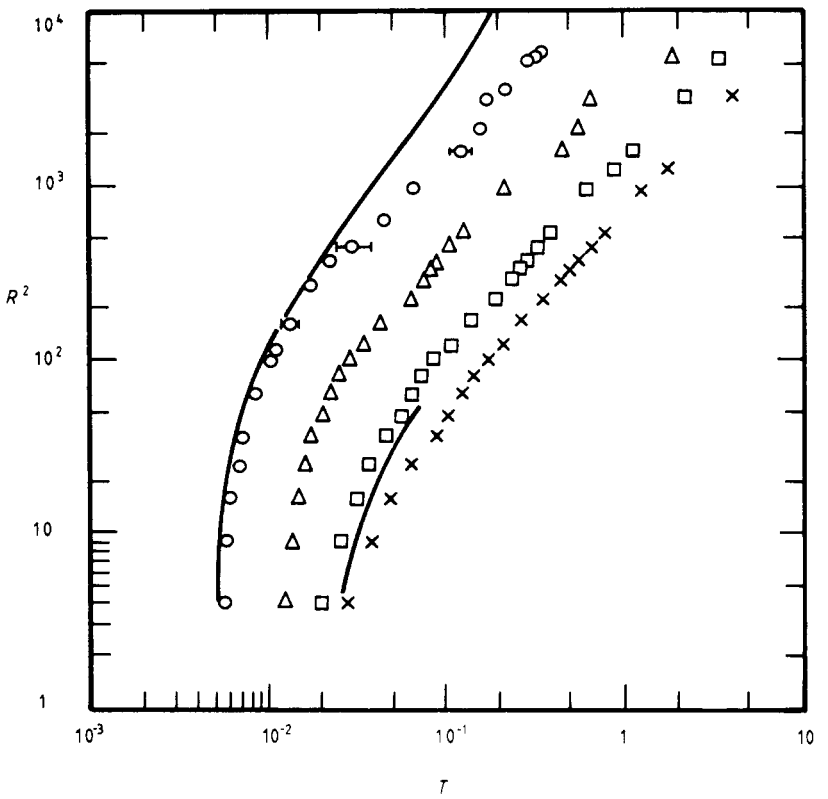

Figure 2. Graph of T as a function of R^2 for the original algorithm of Adler *et al* (1985). R is the distance travelled from the entry point $x = 0$. The lower bounds on the first exit time are indicated as a continuous curve for $p = 0.995$ and $p = 0.975$. Some typical error bars are shown. Note that $R = r + 1$. The circles show the results for $p = 0.995$, the triangles for $p = 0.990$, the squares for $p = 0.9825$ and the crosses for $p = 0.975$.

replaces figure 5 of Adler *et al* (1985) since it contains more data points and better statistics for small values of R^2. The lower bounds made up of contributions from C1, C2, C3, D1, D2 and D3 configurations are indicated for some p and R values. They are extremely close to the Monte Carlo results for small r and p close to 1, where the C2 contributions dominate.

A graph of first exit times as a function of r^2 for the new algorithm is presented in figure 3. Here we observe that the bounds are extremely close to the Monte Carlo results for $10 < R^2 < 400$ and are actually slightly above the data points in a few cases (this is probably due to scatter in the data).

We note that both the bounds and the data for the new algorithm presented in figure 3 are much smaller than the bounds and data for the old one in the region below $R^2 \sim 1000$ for large p. The bounds and data for the new algorithm are consistent with $R^2 \propto T$ for all R, t and those of the old algorithm are consistent with $R^2 \propto 1 - p$ for small R, T and large p. The departure of the bounds for the data for $R^2 > 1000$ in both cases is due to the contributions from terms with three or more normal sites, which begin to be significant in this region.

In figure 4 we present a graph of $D \sim \langle r^2 \rangle / t$ against $p_c - p$. We find that $D \propto (1-p)^{-2}$ as predicted in § 2.

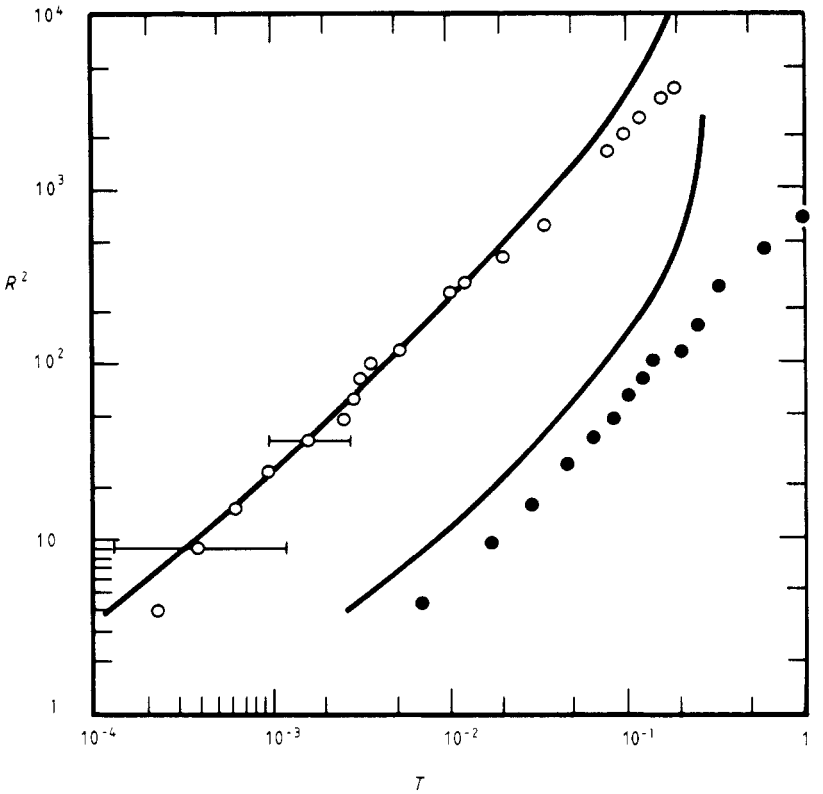

Figure 3. Graph of T as a function of R^2 for the new algorithm. The open circles show the results for $p = 0.995$ and the full circles for $p = 0.975$. R is the distance travelled from the entry point $x = 0$. The lower bounds on the first exit time are indicated as a continuous curve for $p = 0.995$ and $p = 0.975$. Some typical error bars are shown. Note that $R = r + 1$.

5. Scaling

General scaling theories, not only for one dimension, have been discussed in the past (Coniglio and Stanley 1984, Bunde *et al* 1985, Adler *et al* 1985, Hong *et al* 1985) with conflicting results. We feel that equation (6.1) of Hong *et al* is the most plausible one, leading to two different characteristic scaling times diverging as $\tau_A(p_c - p)^{-(2\nu+\mu-\beta)}$ and $\tau_B(p_c - p)^{-(2\nu-s-\beta)}$ respectively, where β is the exponent for the size of the infinite cluster, ν that for the connectivity length, μ that for the conductivity of random resistor networks ($\tau_A = 1$, $\tau_B = \infty$) and s that for the conductivity of random superconducting networks ($\tau_A = 0$, $\tau_B = 1$). We take σ_B/σ_A to be much smaller than $(p_c - p)^{\mu+s}$ and assume $p \leqslant p_c$.

For times much larger than these two characteristic times we expect 'termite diffusion', i.e. $r^2 \propto (t/\tau_B)(p_c - p)^{-s}$; for times much smaller than these times, we expect 'anomalous ant diffusion',

$$r^2 \propto (t/\tau_A)^{(2\nu-\beta)/(2\nu+\mu-\beta)} \tag{15}$$

and for intermediate times we expect a plateau, $r^2 \propto (p_c - p)^{\beta-2\nu}$. No anomalous termite diffusion is predicted, in contrast to an assertion of Adler *et al* (1985) (see also Coniglio and Stanley 1985). Their expectation, however, was based on an erroneous scaling

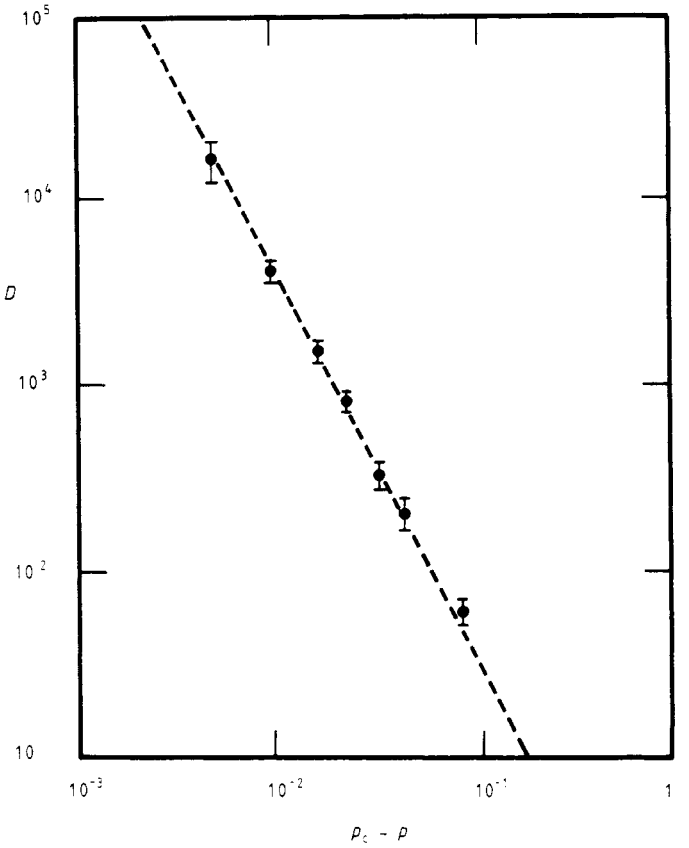

Figure 4. Graph of D as a function of $p_c - p$.

assumption: in their equation (7) the second argument must be multiplied by a frequency factor proportional to σ_A in order to be dimensionally correct. After this correction, their scaling assumption agrees with that of Hong *et al* (1985), whereas their equations (3) and (4) are no longer valid.

For one dimension $\mu = \beta = 0$ and $s = \nu = 1$. These formulae give $r^2 \propto (t/\tau_B) \times (p_c - p)^{-1}$ after sufficiently many steps. But due to the peculiarities of one dimension, with $p_c = 1$, an additional factor of $1 - p = p_c - p$ appears in the short-circuit model as discussed in equation (6). Then the diffusivity r^2/t varies as $(1-p)^{-2}$, as also observed in our Monte Carlo simulations for the short-circuit model. This complication does not arise in the general two-time model, as discussed in equation (11), where the diffusivity for long times diverges as $(p_c - p)^{-1}$ only. Thus these aspects of the scaling theory, as applied to one dimension, agree with our analytic and numerical results.

In summary, we have analysed the various one-dimensional diffusion models capable of representing the random normal superconductor mixture. We have seen that, because one-dimensional percolation has a transition at $p_c = 1$, the Einstein relation linking the conductivity to the diffusion coefficient gives qualitatively different results, depending on whether the charges are confined to the normal metal or not. Further, the small-time behaviour of those models was analysed exactly and found to depend sensitively on extremely detailed information on the model.

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References

Adler J, Aharony A and Stauffer D 1985 *J. Phys. A: Math. Gen.* **18** L129
Bunde A, Coniglio A, Hong D C and Stanley H E 1985 *J. Phys. A: Math. Gen.* **18** L137
Coniglio A and Stanley H E 1984 *Phys. Rev. Lett.* **52** 1068
de Gennes P G 1980 *J. Physique Colloq.* **41** C3-17
Hong D C, Stanley H E, Coniglio A and Bunde A 1985 *Phys. Rev.* B submitted
Sahimi M and Siddiqui S 1985 *J. Phys. A: Math. Gen.* **18** L727